Energetics of particle collisions near dirty rotating extremal black holes:

Banados-Silk-West effect versus Penrose process

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If two particles collide near the horizon of a rotating extremal black hole, under certain conditions the energy $E_{c.m.}$ in the center-of-mass frame can grow without limit (the so-called Banados-Silk-West effect). We consider collisions that produce two other particles. We show that for a generic dirty (surrounded by matter) black hole, there exist upper bounds on the energy and mass of product particles which can be detected at infinity. As a result, the positive energy gain is possible but is quite modest. It mainly depends on two numbers in which near-horizon behavior of the metric is encoded. The obtained results suggest astrophysical limits on the possibility of observation of the products of the collisional Banados-Silk-West effect. These results are consistent with recent calculations for the Kerr metric, extending them to generic dirty black holes. It is shown that for dirty black holes there are types of scenarios of energy extraction impossible in the Kerr case..

PACS numbers: 04.70.Bw, 97.60.Lf

I. INTRODUCTION

Recently, the effect of unbound energy in the centre of mass frame of particles colliding near black holes was discovered [1] ((called the Banados-Silk-West (BSW) effect after the names of its authors).). This effect is interesting from the theoretical point of view since new physics can come into play in the vicinity of black holes at the Planck scale and beyond it,

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with new channels of reactions between particles. From the other hand, there are potential astrophysical manifestations of the BSW effect. Some of them can take place in the vicinity of the horizon, i.e. just the region where this effect occurs. This includes, for example, physics of accretion disks and behavior of extreme mass-ratio inspirals [2] - [4].

Meanwhile, there are also observations on Earth which could be supposedly interpreted on the basis of the BSW effect. In particular, it was conjectured that ultra-high energy cosmic rays detected by the AUGER group [5] might be created in the active galactic nuclei due to the BSW effect in the vicinity of the central supermassive black hole [6] - [11]. Quite recently, another possible astrophysical manifestation of the BSW effect connected with the processes with neutralino was discussed in [12]. For such kind of observations, it is important to know what masses and energies of particles can be detected at infinity. To this end, in [13], [14] the emergent flux emitted by dark matter spikes around intermediate-mass black hole was evaluated from the region close to the horizon. We choose a different approach and consider elementary acts of collisions when a pair of particles is converted into a pair of two other ones. We consider the process in the immediate vicinity of the horizon only that enables us to analyze the problem from the first principles.

The first predictions of such a kind were made in [15] where the authors claimed the existence of the bound on the ratio E/m, where E is the energy of the particle detected at infinity and m is the mass of infalling particles. The result was criticized in [6] where it was noticed that the deviation of [15] was made neglecting the difference between the time-like four-velocity vector and the light-like horizon generators. However, the numeric example suggested in [6] for the case of the Kerr metric as a counter-example to [15], applies to the collision not on the horizon but at some point outside its immediate vicinity, so it cannot be related to the BSW effect directly. Therefore, the issue discussed in [6] remained incomplete and more general treatment is needed to conclude whether or not the bounds on the energy of products of the BSW effect exist.

Quite recently, the problem was considered anew and it was pointed in works [7] - [9] that upper limits on the energy of particles detected at infinity do exist. This was done in [7], [9] for the Kerr metric. In [8], the problem was studied for generic "dirty" (surrounded by matter) black holes but, in particular, one important scenario was overlooked. The aim of the present paper is to suggest a general complete analysis of the reaction between two particles in the context of the BSW process and derive bounds on the energy and mass

of particles escaping to infinity applicable to generic dirty black holes. The last point is especially important since in real astrophysical conditions matter is always present near the horizon of a black hole.

The BSW effect was originally discovered for extremal black holes [1]. It exists also for nonextremal black holes [6], but requires special conditions like multiple scattering (see also [17] for generalization). To avoid these subtleties not connected with the issue under discussion directly, in the present paper we restrict ourselves to extremal black holes. While analyzing the products of the BSW effect detected at infinity, we make focus on the question whether or not one can gain more energy than it was injected. In other words, we discuss the possibility and the limitations of the Penrose process [18] in such situation.

The BSW effect implies that both colliding particles move towards a black hole and, additionally, parameters of one particle are fine-tuned. It should not be confused with a more simple effect which arises when one of the colliding particles moves towards the horizon and the other one moves away from it [19]. We call the latter the Piran and Shanam (PS) effect, this word is used here in the sense of physical effect irrespective of its relevance or irrelevance in practical astrophysics [20]. In the PS effect, the energy in the centre of mass frame grows unbounded just due to the blue-shift of energy, without any fine-tuning. Kinematically, in the case of the PS process, particles experience head-on collision with at least one of them having the speed almost equal to that of light in the frame of a stationary observer. As a result, the relative velocity also tends to the speed of light and the corresponding energy grows indefinitely. Near rotating black holes, the PS effect can lead to the Penrose process (details of the Penrose process in the background of charged and rotating black holes can be found in [22]). From the other hand, for the BSW effect to occur, both particles have to approach the horizon and special conditions are required to achieve the relative velocity which would approach the speed of light [23]. There is also difference between the BSW and PS effects in geometric terms [21].

In the present paper, we combine the approaches of [9] and [8], thus generalizing the results of [7] and [9], derived for the Kerr metric, to a generic dirty rotating extremal black hole. The details of this presentation run along the lines of [9] and at each step the emphasis is made on features that are absent in the Kerr case and arise due to "dirtiness" of black holes.

The general structure of the paper is the following. In Sec. II, we write down the

conservation laws of the radial and angular momenta and energy in the act of collision. We also trace how this is related to the expansion of the four-velocity in terms of the local null tetrad. Division of all particles to two classes (critical and usual) is introduced which is crucial for what follows. In Sec. III, the general conditions are formulated under which a particle can escape to infinity. In Sec. IV, we write down the expansions of the metric coefficient and radial momenta near the horizon in terms of the lapse function N, which is a small quantity near the horizon. This is done separately for usual and (near)critical particles. In Sec. V, general classification of scenarios of collision is suggested according to the kind of particles (usual or critical) and the direction of their motion immediately after collision. In Sec. VI, the conservation of radial momentum is analyzed in the first order in N using the results of Sec. IV. In Sec. VII the same procedure is carried out with terms of the order N^2 taken into account. In Sec. VIII, all allowed scenarios of collision are analyzed and the upper bounds on the mass and energy of particles escaping to infinity are obtained for each scenario separately. This is obtained from the conditions of escaping derived in Sec. III and the conservation law for the radial momentum analyzed in Sec. VI and VII. In Sec. IX, these results are used to elucidate whether energy extraction is possible. Again, all the scenarios are analyzed. In Sec. X, some concrete types of reactions are considered as illustration. The main results are summarized in Sec. XI. In two Appendices we list some technical details used in the main text.

We use units in which fundamental constants $G=c=\hbar=1$.

II. EQUATIONS OF MOTION AND CONSERVATION LAWS

Let us consider the axially symmetric black hole metric

$$ds^2 = -N^2 dt^2 + g_{\phi\phi}(d\phi - \omega dt)^2 + dn^2 + g_{zz} dz^2, \tag{1}$$

where the metric coefficients do not depend on t and ϕ. We want to write down the conservation laws in the act of collision for particles moving in this background. First of all, we must relate the characteristics of each particle to the properties of the metric. Let a particle having the four-velocity u^μ move in this background. It is convenient to expand u^μ with respect to the null tetrad basis. We can write

$$u^\mu = \frac{l^\mu}{2\alpha} + \beta N^\mu + s^\mu \tag{2}$$

where l^μ and N^μ are null vectors normalized according to $l_\mu N^\mu = -1$, s^μ is a space-like vector orthogonal to l^μ and N^μ, $s^\mu = s_a a^\mu + s_b b^\mu$ where a^μ and b^μ are unit vectors orthogonal to each other and to l^μ and N^μ. Then, we can choose

$$l^\mu = (1, N, \omega, 0), \tag{3}$$

$$N^\mu = \frac{1}{2}(\frac{1}{N^2}, -\frac{1}{N}, \frac{\omega}{N^2}, 0) \tag{4}$$

where $x^\mu = (t, n, \phi, z)$.

One can check that the decomposition of the metric

$$g_{\alpha\beta} = -l_\alpha N_\beta - l_\beta N_\alpha + a_\alpha a_\beta + b_\alpha b_\beta \tag{5}$$

is satisfied with

$$a^\mu = (0, 0, \frac{1}{\sqrt{g_{\phi\phi}}}, 0), \tag{6}$$

$$b^\mu = \frac{1}{\sqrt{g_{zz}}}(0, 0, 0, 1),\ b^\mu b_\mu = a^\mu a_\mu = 1, \tag{7}$$

$$\alpha = \frac{\beta}{\delta},\ \delta = 1 + s^2,\ s^2 = s^\mu s_\mu. \tag{8}$$

We assume that the metric is symmetric with respect to z and restrict ourselves by the motion within the equatorial plane. Then,

$$s_a \equiv u_\mu a^\mu = \frac{L}{\sqrt{g}},\ s_b \equiv u^\mu b_\mu = 0,\ s^2 = \frac{L^2}{m^2 g}, \tag{9}$$

where $L = m u_\phi$ is angular momentum per unit mass that conserves due to the independence of the metric on ϕ. For shortness, we use the notation $g \equiv g_{\phi\phi}$. In a similar way, u_0 is conserved because of the independence of the metric on t, $E = -m u_0$ being the energy. Expressing u^μ in terms of u_μ and using (2) and the normalization conditions $u^\mu u_\mu = -1$, one obtains that

$$m u^0 = \frac{X}{N^2}, \tag{10}$$

$$m u^3 = \frac{L}{g} + \frac{\omega X}{N^2} \tag{11}$$

and

$$m u^1 = \varepsilon \frac{Z}{N}, \tag{12}$$

$$X = E - \omega L,,\ Z = \sqrt{X^2 - N^2(m^2 + \frac{L^2}{g})}, \tag{13}$$

where $\varepsilon = +1$ for an outgoing particle and $\varepsilon = -1$ for an ingoing one.

Then, it is easy to calculate

$$\frac{1}{2\alpha} = -u_{\mu} N^{\mu},\ \alpha = \frac{mN^2}{X + \varepsilon Z},\ \beta = -u_{\mu} l^{\mu} = \frac{X - \varepsilon Z.}{m}. \tag{14}$$

One can check that the coefficients α and β obey Eq.(8). We also assume the forward in time condition $u^0 > 0$ as usual, so $X > 0$ everywhere except, possibly, on the horizon where $X = 0$ is also allowed. It is seen from (13) that $Z \leq X$. Therefore, α, $\beta \geq 0$. If the horizon value X_H is positive we call a particle usual. If $(X)_H = 0$ it is called critical (near-critical if $(X)_H$ is small).

Equating the coefficients at l_{μ}, N_{μ} and a^{μ} we obtain for the reaction when two initial particles turn into particles with masses m_3 and m_4:

$$m_1\beta_1 + m_2\beta_2 = m_3\beta_3 + m_4\beta_4, \tag{15}$$

$$\frac{m_1}{\alpha_1} + \frac{m_2}{\alpha_2} = \frac{m_3}{\alpha_3} + \frac{m_4}{\alpha_4}, \tag{16}$$

$$L_1 + L_2 = L_3 + L_4. \tag{17}$$

Equivalently, one can find from (15) - (17) that

$$E_1 + E_2 = E_3 + E_4, \tag{18}$$

$$X_1 + X_2 = X_3 + X_4, \tag{19}$$

$$\varepsilon_1 Z_1 + \varepsilon_2 Z_2 = \varepsilon_3 Z_3 + \varepsilon_4 Z_4 \tag{20}$$

where Eq. (20) has the meaning of the conservation of the radial momentum.

It is worth stressing that individual energies E_i are finite. It is the energy in the centre of mass frame $E_{c.m.}$ which is divergent if the BSW effect takes place (see, e.g. [17] for details).

III. CONDITIONS OF ESCAPING TO INFINITY

As $u^1 \sim Z$ according to (12), the zeros of Z give us the turning points. The condition $Z = 0$ can be rewritten as

$$l^2 - 2\frac{\omega l e}{(\omega^2 - \frac{N^2}{g})} + \frac{(e^2 - N^2)}{(\omega^2 - \frac{N^2}{g})} = 0,\ e = \frac{E}{m},\ l = \frac{L}{m}, \tag{21}$$

whence its roots equal

$$l_{\pm} = \frac{e \pm N\sqrt{Y}}{(\omega - \frac{N^2}{g\omega})}, \, Y = \frac{e^2 + g_{00}}{g\omega^2}, \, g_{00} = -N^2 + g\omega^2. \tag{22}$$

On the horizon,

$$L_{\pm} \equiv ml_{\pm} = \frac{E}{\omega_H} = L_H. \tag{23}$$

The allowed region of motion corresponds to $Z^2 \geq 0$ where Z is given by Eq. (13). The region between turning points is forbidden. We will be interested in the situation when a particle (denoted as particle 3) escapes to infinity from the immediate vicinity of the horizon. This is possible in 2 cases that generalizes the corresponding situation in the Kerr metric [9].

a) $E_3 > m_3$, $L_3 < L_H$, $\varepsilon_3 = +1$. b) $E_3 \geq m_3$, $L_H < L_3 < L_-(E_3)$, $\varepsilon_3 = +1$ or $\varepsilon_3 = -1$. The condition $\varepsilon_3 = -1$ means in this context that particle 3 is moving inward, approaches the outer turning point and bounces back in the outward direction. We consider all these types of scenario in the vicinity of the horizon where $N \ll 1$.

It is convenient to write

$$L = \frac{E}{\omega_H}(1 + \delta). \tag{24}$$

Then, in case (a)

$$\delta < 0. \tag{25}$$

In case (b)

$$\delta \geq 0 \tag{26}$$

but it is bounded from above. Indeed, forward in time condition $X = E - \omega L > 0$ gives us

$$\delta < \frac{\omega_H - \omega}{\omega}. \tag{27}$$

IV. NEAR-HORIZON EXPANSIONS

In this region, the lapse function N is a small quantity and the expansion of the coefficient ω near the extremal horizon takes the general form

$$\omega = \omega_H - B_1 N + B_2 N^2 + O(N^3) \tag{28}$$

where ω_H is the horizon value of ω and B_i is some model-dependent coefficient [24]. Also, we will use the expansion for the metric coefficient g

$$g = g_H + g_1 N + g_2 N^2 + O(N^3). \tag{29}$$

For what follows, we need also the expansions for the quantity Z. This can be found separately for different kinds of particles.

A. Usual particle

For such a particle, $X_H \neq 0$, so we obtain

$$Z = X - \frac{1}{2X}(m^2 + \frac{L^2}{g_H})N^2 + O(N^3) \tag{30}$$

where

$$X = X_H + B_1 L N - B_2 L N^2 + ... \tag{31}$$

B. Critical particle

Now, $X_H = 0$, $L = \frac{E}{\omega_H}$, so

$$X = \frac{E}{\omega_H}(\omega_H - \omega) = \frac{EN}{h}(b - b_2 N) + O(N^3) \tag{32}$$

$$Z = \sqrt{E^2 \frac{(b^2-1)}{h^2} - m^2} N + \frac{E^2}{h^2} \frac{(\frac{g_1}{2g_H} - b b_2)}{\sqrt{E^2 \frac{(b^2-1)}{h^2} - m^2}} N^2 + O(N^3), \tag{33}$$

where we introduced useful notations

$$b = B_1 \sqrt{g_H},\ h = \omega_H \sqrt{g_H},\ b_2 = B_2 \sqrt{g_H}. \tag{34}$$

Hereafter, we assume that $B_1 > 0$ to satisfy the forward in time condition $X > 0$.

C. Near-critical particle

Let us consider a particle which is not exactly critical but, rather, near-critical. It has the angular momentum (24) with $\delta \ll 1$. Then, it follows from (27) and (28) that

$$\delta < bN + N^2(b^2 - b_2) + O(N^3). \tag{35}$$

On the horizon, $(X_3)_H = -\frac{E_3}{\omega_H}\delta$.

Near the horizon, we can take δ that would adjust to small value of N and write the expansion

$$\delta = C_1 N + C_2 N^2 + ... \tag{36}$$

Then,

$$X = NE(\frac{b}{h} - C_1) + QEN^2 + O(N^3), \tag{37}$$

$$Q = C_1\frac{b}{h} - \frac{b_2}{h} - C_2, \tag{38}$$

$$Z = N\sqrt{E^2[(\frac{b}{h} - C_1)^2 - \frac{1}{h^2}] - m^2} + \tau N^2 + O(N^3), \tag{39}$$

$$\tau = \frac{E^2(\rho + \frac{1}{2h^2}\frac{g_1}{g_H})}{\sqrt{E^2[(\frac{b}{h} - C_1)^2 - \frac{1}{h^2}] - m^2)}}, \tag{40}$$

$$\rho = -C_1^2\frac{b}{h} + C_1C_2 + C_1\left(\frac{b^2-1}{h^2} + \frac{b_2}{h}\right) - C_2\frac{b}{h} - \frac{bb_2}{h^2}. \tag{41}$$

D. Comparison with the case of the extremal Kerr black hole

For what follows we need the corresponding quantities for the simplest case of the Kerr extremal metric. This will enable us to compare the general formulas with the results of [7], [9]. We are considering the plane $\theta = \frac{\pi}{2}$ where for the Kerr metric

$$N^2 = \frac{(r-M)^2}{r^2 + M^2 + \frac{2M^3}{r}}, \tag{42}$$

$$\omega = \frac{2M^2}{r^3 + M^2 r + 2M^3}, \tag{43}$$

$$g = r^2 + M^2 + \frac{2M^3}{r}. \tag{44}$$

Then, in the expansions (28), (29)

$$\omega_H = \frac{1}{2M},\ B_1 = \frac{1}{M},\ B_2 = \frac{1}{2M},\ g_H = 4M^2,\ g_1 = 0. \tag{45}$$

$$b = 2,\ h = 1,\ b_2 = 1. \tag{46}$$

In [9] the authors used the expansion in the form $\delta = \delta_1\varepsilon + \delta_2\varepsilon^2 + ...$ where near the horizon $r = \frac{M}{1-\varepsilon}$ with $\varepsilon \ll 1$.

One should bear in mind that the quantity $N \approx \frac{\varepsilon}{2} + \frac{\varepsilon^2}{2} + O(\varepsilon^3)$. Correspondingly,

$$C_1 = 2\delta_1,\, C_2 = 4\delta_2 - 4\delta_1. \tag{47}$$

Then,

$$\rho = -2 + 8(2\delta_1 - \delta_2 - 2\delta_1^2 + \delta_1\delta_2). \tag{48}$$

Eq. (35) turns into $\delta < \varepsilon + \frac{7}{4}\varepsilon^2 + O(\varepsilon^3)$ in agreement with Eq. (3.11) of [9].

V. ALLOWED SCENARIOS OF COLLISIONS

We assume that particles 1 and 2 move towards the horizon, so $\varepsilon_1 = \varepsilon_2 = -1$. The BSW effect is possible only if one of them is critical whereas the other one is usual. Let particle 1 be critical. Then, we must consider different situations with ε_3 and ε_4 using the momentum conservation (20). Some information can be extracted from the calculation of the left and right hand sides of that equation on the horizon. The left hand side of (20) is negative there.

1) $\varepsilon_3 = \varepsilon_4$ It follows from (19) that the right hand side of (20) is equal to $\varepsilon_3 \left(X_2\right)_H$. As $\left(X_2\right)_H > 0$, we must have $\varepsilon_3 = \varepsilon_4 = -1$.

If both particles 3 and 4 are critical, Eq. (19) is not satisfied on the horizon since the right hand side is equal to zero whereas the left hand side is not. One can also see that they cannot be both usual. This follows from the asymptotic behavior (30) and (33) for usual and critical particles. Namely, there are terms of the order N in the left hand side which do not have counterparts in the right hand side. The conclusion is that either particle 3 is critical, particle 4 is usual or vice versa. Below, neglecting in the main approximation terms containing δ and N we obtain the following possibilities.

2) $\varepsilon_3 = -\varepsilon_4$, Then, it follows from (19) that the right hand side of Eq. (20) is equal to

$$\varepsilon_3[2\left(X_3\right)_H - \left(X_2\right)_H]. \tag{49}$$

a) $\varepsilon_3 = 1$, $\varepsilon_4 = -1$. Comparing with the left hand side of (19) we obtain that

$$\left(X_3\right)_H = 0, \tag{50}$$

so particle 3 is critical. Then, (19) also tells us that

$$(X_4)_H = (X_2)_H > 0, \tag{51}$$

so particle 4 is usual.

b) $\varepsilon_3 = -1$, $\varepsilon_4 = +1$. In the same manner, we obtain that

$$(X_3)_H = (X_2)_H > 0, \tag{52}$$

$$(X_4)_H = 0, \tag{53}$$

so particle 3 is usual, particle 4 is critical.

Thus in the pair of particles 3 and 4 it is just the particle escaping to infinity which is critical. For definiteness, we assume that it is particle 3. Then, it follows from the previous analysis that

$$\varepsilon_1 = \varepsilon_2 = \varepsilon_4 = -1 \tag{54}$$

in all cases. If collision occurs not exactly on the horizon but in its vicinity (as it happens for relevant scenarios - see below), particle 3 is not precisely critical but near-critical, so Eq. (24) holds for it. Then, it follows from (19) that in the main approximation (51) is satisfied.

Now, we can apply the near-horizon expansion to different scenarios of escaping. In case (a), Eqs. (25) and (36) give us

$$C_1 < 0. \tag{55}$$

In case (b), we must take into account the presence of the turning point outside the horizon. Then, expanding (22) and neglecting the terms of the second order and higher, we obtain that

$$0 \leq C_1 \leq (C_1)_m = \frac{b}{h} - \sqrt{\frac{m_3^2}{E_3^2} + \frac{1}{h^2}}. \tag{56}$$

The scenarios in which a near-critical particle has $\varepsilon_3 = -1$ immediately after collision and thus moves inward will be called IN scenarios for shortness. If after collision $\varepsilon_3 = +1$ we will call it "OUT" scenario. In turn, we will add "$-$" if $\delta < 0$ and "$+$" if $\delta \geq 0$. In other words, we enumerate possible types of scenarios characterizing them by signs of two quantities - ε and δ. In general, there are 4 combinations: OUT$-$, OUT+, IN+ and IN-. However, it follows from (26) that the scenario IN$-$ should be rejected. For the Kerr metric, the types OUT$-$ and OUT+ are uninteresting since they do not allow energy extraction [7], [9]. However, this is not necessarily so for dirty black holes, so we must discuss all the three remaining types of scenarios.

VI. CONSERVATION OF MOMENTUM IN FIRST ORDER

Now, we have the situation in which particle 1 is critical, particles 2 and 4 are usual, particle 3 is near-critical. Equating terms of the first order in N in (20), and using (30), (32), (33), (37), (39), (40),(54) we obtain

$$F\equiv A_{1}+E_{3}(C_{1}-\frac{b}{h})=\varepsilon_{3}\sqrt{E_{3}^{2}[(\frac{b}{h}-C_{1})^{2}-\frac{1}{h^{2}}]-m_{3}^{2}} \tag{57}$$

where

$$A_{1}=\frac{E_{1}b-\sqrt{E_{1}^{2}(b^{2}-1)-m_{1}^{2}h^{2}}}{h}. \tag{58}$$

Taking here the square of (57) we get

$$C_{1}=\frac{b}{h}-\frac{A_{1}^{2}+m_{3}^{2}+\frac{E_{3}^{2}}{h^{2}}}{2E_{3}A_{1}} \tag{59}$$

whence

$$C_{m}-C_{1}=\frac{\left(A_{1}-\sqrt{m_{3}^{2}+\frac{E_{3}^{2}}{h^{2}}}\right)^{2}}{2E_{3}A_{1}}\geq 0. \tag{60}$$

After the substitution of (59) back into (57), we obtain that

$$F=\frac{A_{1}^{2}-m_{3}^{2}-\frac{E_{3}^{2}}{h^{2}}}{2A_{1}}. \tag{61}$$

For $E_{1}\geq m_{1}$ it follows from (58) that

$$b-\sqrt{b^{2}-1}\leq\frac{A_{1}h}{E_{1}}\leq b-\sqrt{b^{2}-1-h^{2}}. \tag{62}$$

VII. CONSERVATION OF MOMENTUM IN SECOND ORDER

If we equate the terms of order N^{2} in (20) taking into account (30), (33), (39), (51) we obtain

$$Y_{L}=Y_{R} \tag{63}$$

where Y_{L} and Y_{R} are corresponding coefficients at N^{2} in $Z_{2}-Z_{4}$ and $-Z_{1}-\varepsilon_{3}Z_{3}$, respectively. Straightforward calculations gives us

$$Y_{L}\equiv\frac{1}{2\left(X_{2}\right)_{H}}\{m_{4}^{2}-m_{2}^{2}+\frac{(E_{1}+E_{2}-E_{3})^{2}-E_{2}^{2}+2\left(X_{2}\right)_{H}(b_{2}h-1)(E_{1}-E_{3})}{h^{2}}\}+E_{3}(\frac{b}{h}C_{1}-C_{2}) \tag{64}$$

and

$$Y_R \equiv (\frac{bb_2}{h^2} - \frac{g_1}{2g_H h^2})\frac{E_1^2}{\sqrt{E_1^2\frac{(b^2-1)}{h^2} - m_1^2}} - \varepsilon_3\tau, \tag{65}$$

τ is given by Eqs. (40) and (41).

In the Kerr case using (45) - (48) one can see that (64) and (65) lead to Eq. (4.14) of [9] for $\varepsilon_3 = -1$.

VIII. GENERAL BOUNDS ON ENERGY AND MASS

A. Scenario IN+

Now, $\varepsilon_3 = -1$. Then, it follows from (57), (61) that

$$E_3^2 \geq \lambda_0^2, \tag{66}$$

$$\lambda_0^2 \equiv h^2(A_1^2 - m_3^2). \tag{67}$$

The condition $C_1 \geq 0$ in (59) gives rise to inequality

$$E_3^2 - 2E_3bhA_1 + h^2(A_1^2 + m_3^2) \equiv (E_3 - \lambda_+)(E_3 - \lambda_-) \leq 0, \tag{68}$$

so

$$\lambda_- \leq E_3 \leq \lambda_+ \tag{69}$$

where

$$\lambda_\pm = h[A_1 b \pm \sqrt{A_1^2(b^2-1) - m_3^2}]. \tag{70}$$

It follows from (62) that the roots satisfy the inequalities

$$(b - \sqrt{b^2-1})b \leq \frac{\lambda_+}{E_1} \leq \left(b - \sqrt{b^2 - 1 - h^2}\right)(b + \sqrt{b^2-1}), \tag{71}$$

$$(b - \sqrt{b^2-1})^2 \leq \frac{\lambda_-}{E_1} \leq (b - \sqrt{b^2-1-h^2})b. \tag{72}$$

As the roots $\lambda_\pm$ should be real in the case under discussion, (70) entails

$$m_3 \leq m_a \equiv A_1\sqrt{b^2-1}. \tag{73}$$

The condition $m_3 \leq E_3 \leq \lambda_+$ leads to $m_3 \leq \lambda_+$ whence

$$m_3 \leq m^* = A_1 \frac{h}{1+h^2}(b + \sqrt{b^2 - 1 - h^2}). \tag{74}$$

Although we have also the bound (73), it is easy to show that $m^* < m_a$. Therefore, the bound (74) is more accurate.

Taking into account (58) we have

$$m_3 \leq m_B \equiv \frac{1}{1+h^2}(b + \sqrt{b^2 - 1 - h^2}) \left(E_1 b - \sqrt{E_1^2(b^2-1) - m_1^2 h^2} \right). \tag{75}$$

To have the upper bound for E_3, we use (69) and the fact that

$$\lambda_+ \leq h A_1 [b + \sqrt{(b^2 - 1)}]. \tag{76}$$

Then,

$$E_3 \leq E_B = (b + \sqrt{b^2 - 1}) \left(E_1 b - \sqrt{E_1^2(b^2-1) - m_1^2 h^2} \right). \tag{77}$$

1. *Some properties of the bounds*

It is seen from (75) and (77) that

$$\gamma \equiv \frac{E_B}{m_B} = \frac{(1+h^2)(b + \sqrt{b^2-1})}{b + \sqrt{b^2 - 1 - h^2}} > 1 \tag{78}$$

is a constant.

One can write

$$\frac{m_B}{m_1} = f(x),\ x = \frac{E_1}{m_1},\ f = \frac{1}{1+h^2}(b + \sqrt{b^2 - 1 - h^2}) \left(xb - \sqrt{x^2(b^2-1) - h^2} \right). \tag{79}$$

It is seen that $f(1) = 1$. If $E_1 \gg m_1$ or, equivalently, $x \gg 1$, the function $f \sim x \gg 1$, so $m_B \gg m_1$. It is monotonic if $h \leq \frac{\sqrt{b^2-1}}{b}$. If $\frac{\sqrt{b^2-1}}{b} < h \leq \sqrt{b^2-1}$ this function takes a minimum at

$$x_0 = \frac{bh}{\sqrt{b^2-1}}, \tag{80}$$

$$f(x_0) = \frac{h}{1+h^2} \frac{b + \sqrt{b^2 - 1 - h^2}}{\sqrt{b^2-1}}. \tag{81}$$

It is also instructive to look at the ratio $\frac{m_B}{E_1} = \chi(x)$ where

$$\chi = \frac{1}{1+h^2}(b + \sqrt{b^2 - 1 - h^2}) \left(b - \sqrt{b^2 - 1 - \frac{h^2}{x}} \right) \tag{82}$$

it is seen that $\frac{d\chi}{dx} < 0$. Here, $\chi(1) = 1$ Therefore, $\chi < 1$ for $x > 1$, so $m_B < E_1$. Therefore, the ratio $\frac{E_B}{E_1}$ attains the maximum value for $E_1 = m_1$. Then, $\frac{E_B}{E_1}$ is given by Eq. (78). It is seen from (70) and (77) that $\lambda_+ = E_B$ if and only if $m_3 = 0$.

Let $E_3 = \lambda_+$. If $m_3 \geq A_1$, Eq. (66) is satisfied automatically. Let $m_3 \leq A_1$, $\sqrt{b^2 - 1}A_1$. It is seen from (70) and (66) that

$$\lambda_+ \geq hA_1 b \geq \lambda_0. \tag{83}$$

If $E_3 = \lambda_\pm$, it follows from (59), (68) that $C_1 = 0$.

Now, it is instructive to consider some consequences of Eqs. (58), (70) for massive and massless particles 1 and 3.

Particles 1 and 3 are massless:

$$\lambda_+ = E_1. \tag{84}$$

Particle 1 is massless, particle 3 is massive. Then,

$$A_1 = E_1(\frac{b - \sqrt{b^2 - 1}}{h}),\ \lambda_+ < hA_1(b + \sqrt{b^2 - 1}), \tag{85}$$

so

$$\lambda_+ < E_1. \tag{86}$$

As in the limit $m_1 \ll E_1$, we have $\lambda_+ \leq E_1$, there is no net energy extraction in this case.

Particle 1 is massive, particle 3 is massless. Then,

$$A_1 > E_1(\frac{b - \sqrt{b^2 - 1}}{h}),\ \lambda_+ = hA_1(b + \sqrt{b^2 - 1}), \tag{87}$$

so

$$\lambda_+ > E_1. \tag{88}$$

For the particular case described by Eq. (46), all aforementioned properties agree with those for the Kerr black hole [9].

B. Scenario OUT+

In this scenario, the condition $C_1 \geq 0$ should be satisfied as well as in case IN+ considered before. Therefore, Eq. (69) holds.

Eqs. (57), (61) and the condition $\varepsilon_3 = +1$ give us

$$E_3^2 \leq \lambda_0^2, \tag{89}$$

$$m_3 \leq A_1 \tag{90}$$

instead of (66). Thus we have two upper bounds $E_3 \leq \lambda_0$ and $E_3 \leq \lambda_+$. Because of the property (83), the bound (89) is more relevant, so

$$\lambda_- \leq E_3 \leq \lambda_0. \tag{91}$$

It is seen from (62) that

$$\lambda_0 \leq hA_1 \leq \lambda_1 \equiv E_1(b - \sqrt{b^2 - 1 - h^2}). \tag{92}$$

If $\frac{\lambda_1}{E_1} < 1$, there is no net energy extraction since in this case $E_3 < E_1$. In particular, this happens in Kerr case when $\frac{\lambda_1}{E_1} = 2 - \sqrt{2}$ and this is the reason why this scenario was rejected in [9]. However, for a more general metric, we may try to obtain $\lambda_1 > E_1$ that gives the necessary condition

$$\sqrt{1 + h^2} < b < 1 + \frac{h^2}{2}. \tag{93}$$

Consistency of Eqs. (69) and (89) requires

$$\lambda_- \leq \lambda_0. \tag{94}$$

Then one can find from (72) that

$$(b - \sqrt{b^2 - 1})^2 \leq b - \sqrt{b^2 - 1 - h^2}. \tag{95}$$

One can observe that $(b - \sqrt{b^2 - 1})^2 \leq b - \sqrt{b^2 - 1} \leq b - \sqrt{b^2 - 1 - h^2}$, so (95) is satisfied. Thus the inequality $\lambda_1 > E_1$ is indeed possible, so the case OUT+ is of some potential interest for the energy extraction in contrast to the Kerr case [7], [9],

C. Scenario OUT−

As $\varepsilon_3 = +1$, inequalities (89) and (90) should be still satisfied since they follow from $F \geq 0$, where F is given by (61). Eq. (92) that follows from $E_3 \leq \lambda_0$ is valid as well. As we want to have the possibility of the energy extraction we must assume (93).

This is not the end of story. To check the possibility of such a variant, we must also take into account the condition (55) where C_1 is given by Eq. (59) and verify its compatibility with (89) and (93). Then, from (59) we have that

$$\frac{b}{h} < \frac{A_1^2 + m_3^2 + \frac{E_3^2}{h^2}}{2E_3 A_1}, \tag{96}$$

so that

$$(E_3 - \lambda_+)(E_3 - \lambda_-) > 0. \tag{97}$$

Now, we will consider different possibilities separately.

a) The roots (70) λ_+ and λ_- of (97) are complex,

$$A_1^2(b^2 - 1) < m_3^2 \leq A_1^2, b^2 < 2. \tag{98}$$

As a result, there is no bound on $\frac{E_3}{E_1}$ from (97). However, the bound (89) persists. If (93) is satisfied, then the upper limit $\lambda_1 > E_1$. In addition, it follows from (93) and (98) that $h < 1$.

Let now the roots λ_+ and λ_- be real, so

$$m_3^2 \leq A_1^2(b^2 - 1). \tag{99}$$

b) $E_3 > \lambda_+$.

This case is inconsistent with (89) and (83) and should be rejected.

c) $E_3 < \lambda_-$. If $\frac{\lambda_-}{E_1} < 1$ and $\frac{\lambda_0}{E_1} < 1$, no energy extraction is possible. To avoid this case of no interest, we require that $\lambda_- > E_1$ and $\lambda_0 > E_1$. The first condition requires (93). The second one leads to the bound for the mass:

$$m_3^2 \leq \frac{E_1^2}{h^2}[\left(b - \sqrt{b^2 - 1 - h^2}\right)^2 - 1] \tag{100}$$

where we took into account (62). The right hand side of (100) is nonnegative if (93) is satisfied.

IX. ENERGY EXTRACTION AND UNCONDITIONAL UPPER LIMITS ON ITS EFFICIENCY

It follows from (63) - (65) that

$$m_4^2 + 2\left(X_2\right)_H S = m_2^2 + \frac{2E_2(E_3 - E_1) - (E_1 - E_3)^2 + 2\left(X_2\right)_H (b_2 h - 1)(E_3 - E_1)}{h^2}, \tag{101}$$

$$S = -Y_R + (\frac{b_2}{h}C_1 - C_2)E_3. \tag{102}$$

If $S \geq 0$ (the corresponding conditions are discussed in Appendix B), the left hand side of (101) is positive (or, at lest, nonnegative) and this gives us the constraint on E_2. Since the case $E_1 \geq E_3$ is not interesting, we assume that $E_1 < E_3$. Then, if follows from (101) that

$$E_2 \geq \frac{1}{2}[(E_3 - E_1) - \frac{m_2^2 h^2}{E_3 - E_1}] - \nu \equiv \kappa,\ \nu \equiv (X_2)_H\,(b_2 h - 1). \tag{103}$$

In the Kerr case, $\nu = 0$, $h = 1$ and (103) reduces to Eq. (4.15) of [9].

We want to derive the unconditional upper limit on $\frac{E_3}{E_1}$ using this inequality, when needed. Below, we discuss different scenarios separately.

A. Scenario IN+

To gain the highest efficiency of extraction, we want to have $E_3 = \lambda_+$ which is the maximum possible value according to (69). In this case, according to (59) and (68), $C_1 = 0$, so $\delta \geq 0$ gives us $C_2 \geq 0$. To maximize the possible outcome, we concentrate on the case when $\lambda_+ = E_B$ given by Eq. (77). In turn, this implies that $m_3 = 0$.

The efficiency of the possible energy extraction is given by the quantity

$$\eta = \frac{E_3}{E_1 + E_2}. \tag{104}$$

As usual, we assume that $E_2 \geq m_2$, so a particle is injected from infinity. As we have two conditions $E_2 \geq m_2$ and $E_2 \geq \kappa$ we must consider two relationships between m_2 and κ.

a) $\kappa > m_2$. Then, it follows from (103) that $m_2 < m_+$ where

$$m_+ = \frac{y}{h^2}(\sqrt{1 + h^2 - 2\frac{\nu h^2}{y}} - 1)\,,\ y = \lambda_+ - E_1 > 0. \tag{105}$$

For the efficiency we obtain

$$\eta \leq \eta_m = \frac{\lambda_+}{E_1 + \kappa}. \tag{106}$$

We are interested in the cases when extraction of energy is possible, so $\eta_m > 1$. Using (103) and the fact that now $E_3 = \lambda_+$, we obtain

$$y^2 + 2\nu y + m_2^2 h^2 > 0. \tag{107}$$

If $b_2h - 1 \geq 0$, so $\nu \geq 0$, then inequality (107) is satisfied automatically. In this case, the positivity of the expression inside the radical in (105) implies also

$$\frac{2\nu h^2}{1+h^2} < E_1(\gamma - 1) \tag{108}$$

where γ is given by (78).

If $\nu < 0$, there are different options for Eq. (107) to be satisfied. (i) $m_2h > |\nu|$, then (107) is satisfied for any y, so $\eta_m > 1$ always. In two other cases additional constrains are required: (ii) $y \leq y_-$, so $\lambda_+ \leq E_1 + y_-$, (iii) $y \geq y_+$, so $\lambda_+ \geq E_1 + y_+$ where $y_\pm = |\nu| \pm \sqrt{|\nu|^2 - m_2^2h^2}$ are roots of (107).

b) $\kappa \leq m_2$. Then, $m_2 \geq m_+$,

$$\eta \leq \eta_m = \frac{\lambda_+}{E_1 + m_2}. \tag{109}$$

The condition $\eta_m > 1$ requires $m_2 < \lambda_+ - E_2$.

Now, we want to find some general unconditional upper bound on η. We want to maximize η_m, thus minimizing E_2 as a function of m_2 for given E_1, m_1. As we must have simultaneously $E_2 \geq m_2$ (particle 2 is injected from infinity) and $E_2 \geq \kappa$, this corresponds to $m_2 = \kappa$, whence $m_2 = m_+$ with m_+ given by (105). To get the possible maximum η, we also put $E_1 = m_1$ and $\lambda_+ = E_B$.

As a result, we have

$$\eta_m = \frac{\lambda_+}{m_1 + \kappa}. \tag{110}$$

Now, taking into account (77), (105) we can write

$$\lambda_+ = qm_1,\ q = (b + \sqrt{b^2 - 1})(b - \sqrt{b^2 - 1 - h^2}), \tag{111}$$

$$1 \leq q \leq 1 + h^2 \leq b^2. \tag{112}$$

$$m_+ = sm_1,\ s = \frac{1 - q + \sqrt{(1+h^2)(q-1)^2 - 2\nu h^2(q-1)}}{h^2} \tag{113}$$

$$\eta_m = \frac{\lambda_+}{m_1 + m_2} = \frac{qh^2}{1 + h^2 - q + \sqrt{(1+h^2)(q-1)^2 - 2\nu(q-1)}} \tag{114}$$

These formulas are simplified when $\nu = 0$. Then,

$$s = \frac{q-1}{h^2}\left(\sqrt{1+h^2} - 1\right), \tag{115}$$

$$\eta_m = \frac{q(\sqrt{1+h^2} + 1)}{q + \sqrt{1+h^2}} > 1. \tag{116}$$

It is also seen from (112), (116) that $\eta_m \leq q \leq 1 + h^2 \leq b^2$. When $b \gg 1 + h^2$, one obtains that $q \approx 1 + h^2$ and $\eta_m \approx \sqrt{1 + h^2}$.

In the Kerr case, $\nu = 0$, $q = (2 + \sqrt{3})(2 - \sqrt{2})$, $s = (\sqrt{2} - 1)(q - 1)$,

$$\eta_m = \frac{2(2 + \sqrt{3})}{q + 2} \approx 1.466 \tag{117}$$

that agrees with the results of [7], [9].

B. Scenario OUT +

Now, according to (83), (89),

$$\eta_m = \frac{\lambda_0}{E_1 + E_2} \tag{118}$$

where we put $E_3 = \lambda_0$ instead of $E_3 = \lambda_+$ typical of IN+ case. As usual, we choose $E_1 = m_1$ as a reference point. Apart from this, now we do not have the condition $C_1 = 0$ that followed from $E_3 = \lambda_+$ in the aforementioned scenario. Correspondingly, there is no definite restriction on S in (101) and there is no bound on E_2 similar to (103). Therefore, we put $E_2 = m_2$. To gain the maximum efficiency, we simply choose $m_2 = 0$. Then,

$$\eta \leq \frac{\lambda_0}{m_1} \leq \eta_m = b - \sqrt{b^2 - 1 - h^2}, \tag{119}$$

where (61), (67) where taken into account. Eq. (119) corresponds to $m_3 = 0$ and gives the unconditional limit for the scenario under discussion. It is possible to gain $\eta_m > 1$, provided Eq. (93) is satisfied. This option is absent for the Kerr case $b = 2$, $h = 1$. For a given h, the maximum of η_m is achieved at $b = \sqrt{1 + h^2}$ when $\eta_m = \sqrt{1 + h^2} > 1$.

C. Scenario OUT -

Now, we must enumerate all cases already considered in Sec. VIII C and apply the corresponding results to the evaluation of the extraction efficiency. In doing so, we put $E_1 = m_1$, $m_2 = 0$.

a) Eq. (98) is valid. There is no bound from Eq. (97), so the only bound is $E_3 \leq \lambda_0$. Then, Eq. (119) applies as well as subsequent discussion, so $\eta_m > 1$ is possible. Consistency of (98) and (93) requires also $h < 1$.

b) Eq. (99) is valid. Then, there is competition between two bounds $E_3 \leq \lambda_0$ and $E_3 \leq \lambda_-$. As λ_0 (67) is a monotonically decreasing function of m_3 and λ_- is a monotonically increasing one, $E_3 \leq \lambda^*$ where λ^* corresponds to the point of their intersection. In that point,

$$m_3^2 = A_1^2 \frac{b^2-1}{b^2}, \tag{120}$$

$$\eta_m = \frac{\lambda^*}{m_1} = \frac{b - \sqrt{b^2-1-h^2}}{b} < 1, \tag{121}$$

so there is no energy extraction.

We would like to remind that in IN+ scenario, the massless case $m_3 = 0$ was favorite. This is not so in the scenario under discussion since it is the condition (98) that is consistent with $\eta_m > 1$ but this implies that $m_3 \neq 0$.

X. EXAMPLES OF REACTIONS

In addition to general bounds, we illustrate the efficiency of energy extraction by some simple examples. For simplicity, in all scenarios we take $\nu = 0$.

A. Scenario IN+

1. *Elastic collision*

For simplicity, we choose $m_1 = m_2 = m_3 = m_4 = m_0$, $E_1 = m_0$.

Then, it follows from (58), (70) that

$$\frac{\lambda_+}{m_0} \equiv \mu = b(b - \sqrt{b^2-1-h^2}) + \sqrt{Y^2},\ Y^2 = (b^2-1)(b - \sqrt{b^2-1-h^2})^2 - h^2. \tag{122}$$

If

$$\frac{\sqrt{b^2-1}}{b} \leq h \leq \sqrt{b^2-1}, \tag{123}$$

it turns out that

$$|Y| = b^2 - 1 - \sqrt{b^2-1-h^2}\, b, \tag{124}$$

so

$$\mu = 2b^2 - 1 - 2b\sqrt{b^2-1-h^2} > 1. \tag{125}$$

One can see that

$$1 < \mu < 2b^2 - 1, \tag{126}$$

where the minimum and maximum values correspond to (123). For $b \gg 1 + h^2$,

$$\mu \approx 1 + h^2. \tag{127}$$

For the Kerr case, it follows from (46) that

$$\mu = 7 - 4\sqrt{2} \approx 1.343 \tag{128}$$

in agreement with Eq. (5.1) of [9].

To evaluate the efficiency of the extraction process $\eta = \frac{E_3}{E_1+E_2}$, we take $E_3 = \lambda_+$. From (103), (105) we have now

$$\frac{\kappa}{m_0} = \frac{\mu^2 - 2\mu + 1 - h^2}{2(\mu - 1)}, \tag{129}$$

$$\frac{m_+}{m_0} = \frac{\mu - 1}{1 + \sqrt{1 + h^2}}. \tag{130}$$

a) $\kappa > m_2$. Now, according to (105), $m_2 < m_+$ where now it follows from (130) that we must have

$$\mu - 1 > 1 + \sqrt{1 + h^2}. \tag{131}$$

The maximum value of the left hand side equals $2(b^2 - 1)$, so the necessary condition is $b^2 > \frac{3}{2} + \frac{\sqrt{1+h^2}}{2}$.

Then, the upper limit is given by (106) where κ is taken from (129),

$$\eta_m = \frac{2\mu(\mu - 1)}{\mu^2 - 1 - h^2}. \tag{132}$$

In particular, it follows from (127), (132) that for large b, $\eta_m \approx 2$.

For the Kerr metric, inequality (131) is not satisfied, so this case is absent.

b) $\kappa \le m_2$, $m_2 \ge m_+$. Then, Eq. (109) applies,

$$\eta_m = \frac{\mu}{2}. \tag{133}$$

For the Kerr case $\eta_m \approx 0.672 < 1$, so there is no energy extraction in agreement with Eq. (5.2) of [9]. However, if b is sufficiently large, it is possible to have $\eta_m > 1$. Indeed, for $b \gg 1 + h^2$ one sees from (127) that $\eta_m \approx \frac{1+h^2}{2}$, so for $h > 1$ we obtain that $\eta_m > 1$.

If, instead of (123),

$$h < \frac{\sqrt{b^2 - 1}}{b}, \tag{134}$$

the quantity Y in (124) is such that

$$|Y| = \sqrt{b^2 - 1 - h^2}b + 1 - b^2, \tag{135}$$

$$\frac{\lambda_+}{m_0} = 1, \tag{136}$$

and there is no energy extraction.

Let now $m_1 = m_3 \equiv m_0$, $m_2 = m_4$ but $m_2 \neq m_0$. We can optimize η by taking $m_2 = \kappa = m_+$ where m_+ is given by (130). Then, η_m is given by (106),

$$\eta_m = \frac{\mu(1 + \sqrt{1 + h^2})}{\sqrt{1 + h^2} + \mu} > 1. \tag{137}$$

For large μ, $\eta_m \approx 1 + \sqrt{1 + h^2}$.

In the Kerr case (128) Eq. (137) reduces to

$$\eta_m = \frac{18\sqrt{2} + 11}{13}, \tag{138}$$

so we return to Eq. (5.4) of [9].

2. Compton scattering

To gain the maximum of efficiency, we take $m_3 = 0$ since it is the condition of having $\lambda_+ = E_3 = E_B$. This is explained in discussion after Eq. (82) that generalize the corresponding observations in [7], [9].

1) $m_1 = m_3 = 0$, $m_2 = m_4 = m_0$

Then, it follows from (84) that $\lambda_+ = E_1$,so $C_1 = 0$ according to (59). Eqs. (63) - (65) with $C_1 = 0$ give us that $C_2 = 0$. It means that in the given approximation, particle 3 is critical and one cannot distinguish between particles 1 and 3 at all. Actually, one cannot detect scattering in this approximation, so the situation is similar to that in the Kerr case [9].

2) $m_4 = m_1 = m_0$, $m_2 = m_3 = 0$, $E_1 = m_0$

It follows from (70) that

$$\mu = \frac{\lambda_+}{m_0} = (b + \sqrt{b^2 - 1})(b - \sqrt{b^2 - 1 - h^2}) > 1. \tag{139}$$

Now, $m_2 = 0 < \kappa = \frac{\mu - 1}{2}$. According to (106),

$$\eta_m = \frac{2\mu}{\mu + 1} \tag{140}$$

For sufficiently large b and, hence, large μ the upper limit can approach 2. In the Kerr case,

$$\mu = \left(2+\sqrt{3}\right)(2-\sqrt{2}) \approx 2.186 \tag{141}$$

and

$$\eta_m = \frac{2(54+14\sqrt{3}-10\sqrt{2}+\sqrt{6})}{97} \tag{142}$$

in agreement with Eq. (5.7) of [9].

3. Pair annihilation

Let two massive particles collide to produce two massless ones,

$$m_1 = m_2 = m_0,\, m_3 = m_4 = 0. \tag{143}$$

For definiteness,

$$E_1 = m_0, \tag{144}$$

as usual. Then, we obtain again that (139), (130) and (129) hold. Therefore, for a) $\kappa > m_0$ Eq. (132) is valid, for b) $\kappa \leq m_0$, Eq. (133) applies as well as corresponding discussion. For the Kerr metric, only case b) is realized with (141) and $\eta_m = \frac{\mu}{2} \approx 1.093$, so we return to Eqs. (5.8), (6.9) of [9].

B. Scenario OUT+

1. Elastic collision

Let $m_1 = m_2 = m_3 = m_4 = m_0$, $E_1 = m_0$. Then, it follows from (118) that

$$\eta_m = \frac{\lambda_0}{2m_0}. \tag{145}$$

According to (58), (66),

$$\eta_m = \frac{\left(b-\sqrt{b^2-1-h^2}\right)^2 - h^2}{2} < 1, \tag{146}$$

so there is no energy extraction.

2. Compton scattering

1) $m_1 = m_3 = 0$, $m_2 = m_4 = m_0 = E_1$

Now, η_m is given by $\eta_m = \frac{\lambda_0}{m_0}$. It follows from (58), (66) that

$$\lambda_0 = hA_1 = m_0(b - \sqrt{b^2 - 1}), \tag{147}$$

$$\eta_m = b - \sqrt{b^2 - 1} < 1, \tag{148}$$

so there is no net energy extraction.

2) $m_4 = m_1 = m_0$, $m_2 = m_3 = 0$, $E_1 = m_0$

In a similar manner,

$$\lambda_0 = hA_1 = m_0(b - \sqrt{b^2 - 1 - h^2}), \tag{149}$$

so

$$\eta_m = b - \sqrt{b^2 - 1 - h^2}. \tag{150}$$

One can check that $\eta_m > 1$ if (93) holds.

In particular, for $b = \sqrt{1 + h^2}$, $\eta_m = \sqrt{1 + h^2} > 1$.

3. Pair annihilation

We choose for simplicity

$$m_1 = m_2 = m_0 = E_1,\ m_3 = m_4 = 0. \tag{151}$$

Now, Eq. (149) applies,

$$\eta_m = \frac{b - \sqrt{b^2 - 1 - h^2}}{2}. \tag{152}$$

It is seen that $\eta_m > 1$ if $b^2 < \frac{5+h^2}{4}$.

The cases with $m_4 = 0$ are of no interest since one can check that they give no energy extraction.

C. Scenario OUT –

According to previous analysis, there are no new interesting options here with $\eta_m > 1$, so the situation is similar to the previous case.

XI. SUMMARY AND CONCLUSION

We investigated different types of high-energetic processes near black holes. As the BSW effect means that the energy in the centre of mass frame $E_{c.m.}$ of two colliding particles grows unbound, it would seem that this effect is favorite for observation of high-energy particles at infinity. These intuitive expectations are not confirmed. It turned out that there are bounds on the mass and energy of particles created due to the BSW effect. It is possible in some cases to gain more energy than it was injected but the excess of energy at infinity is restricted. In doing so, we generalized the corresponding properties of the Kerr metric [7], [9].

Meanwhile, there are some new features which exist for dirty black holes only and are absent in the Kerr case. First of all, it concerns the possible choice of scenario of collisions. In the Kerr case, the only potential scenario that gives energy extraction is (in our notations) IN+ when a particle immediately after collision moves inwardly, bounces and only afterwards goes to infinity [19], [7], [9]. Meanwhile, it turned out that in general two other scenarios OUT + and OUT − are also possible, so a particle can escape to infinity directly. The value of energy extraction depends crucially on relationship between just two numbers (b and h) that characterize near-horizon behavior of the metric. Account for generic b and h not only opens new scenarios for energy extraction but also extends significantly the diversity of possible cases within the old scenario IN+ typical of the Kerr metric. For $b \sim h \sim 1$, the maximum possible extraction efficiency η_m may be higher than in the Kerr case being enhanced by numeric factors like 2 or even higher. In some cases, η_m even grows formally without limit but this also requires "exotic" situations with large b or h, so in effect η_m remains limited. Thus in spite of essential extension of the whole picture, the main conclusion does not change radically from the Kerr case. In principle, the corresponding bounds on the energy become weaker but the bounds persist. The results for the unconditional upper limit for different scenarios are summarized in Table 1.

Scenario	$\eta_m > 1$ for unconditional upper limit	Upper bound on E_3
IN+	always (Kerr black hole included)	λ_+
OUT+	$\sqrt{1+h^2} < b < 1 + \frac{h^2}{2}$	λ_0
OUT-	$\sqrt{1+h^2} < b < 1 + \frac{h^2}{2}$, $b < \sqrt{2}$, $h < 1$	λ_0

Table 1: Scenarios with possible extraction of energy and their main properties.

It is seen that scenario IN+ is the most favorable, scenario OUT- is the most restrictive.

One can say that the BSW effect and the Penrose process due to near-horizon collisions are mutually "unfriendly". Thus there exist serious difficulties for direct observation of the consequences of the BSW effect although they become milder if one deals with a dirty black hole instead of the Kerr metric. However, this effect can leave indirect imprint at infinity, since new channels of reactions can be open which are forbidden in the laboratory experiments.

The analysis of elementary acts of collision from the first principles suggested in the present work, can be used also for the analysis of high-energy processes in the vicinity of charged static black holes. This will be done elsewhere.

Acknowledgments

I thank Igor Tanatarov for reading the manuscript and useful comments. This work was supported in part by the Cosmomicrophysics section of the Programme of the Space Research of the National Academy of Sciences of Ukraine.

XII. APPENDIX A: SPECIAL CASE: NEAR-HORIZON CIRCULAR ORBIT

This appendix generalizes the contents of Appendix A in [9]. Formally, in Eq. (65) divergences appear if $E_1^2 = E_0^2 \equiv \frac{m^2 h^2}{b^2-1}$. Actually, this case corresponds to a "circular" orbit with the constant proper distance ($Z_1 = 0$) near the horizon and requires some inessential changes. The circular orbit is defined by equations

$$Z^2 = (E - \omega L)^2 - N^2(m^2 + \frac{L^2}{g}) = 0 \tag{153}$$

and

$$\frac{dZ^2}{dN} = 0. \tag{154}$$

It follows from (153), (154) that

$$E - \omega L = N\sqrt{m^2 + \frac{L^2}{g}} \tag{155}$$

and

$$-\sqrt{m^2+\frac{L^2}{g}}L\frac{d\omega}{dN}-(m^2+\frac{L^2}{g})+N\frac{L^2}{2g^2}\frac{dg}{dN}=0 \tag{156}$$

Near the horizon where $N \ll 1$ one can find from (155) and (156) that

$$E = E_0(1+e_1N+e_2N^2)+O(N^3), \tag{157}$$

$$L=\frac{E_0}{\omega_H}(1+\gamma_1 N+\gamma_2 N^2)+O(N^2). \tag{158}$$

Then, one can obtain from (28) and (156) that

$$\gamma_1 = e_1 = \frac{2b_2b-\frac{g_1}{g_H}}{b^2-1}, \tag{159}$$

$$\frac{\omega_H L}{E}=1+\tilde{\gamma}N^2+O(N^3). \tag{160}$$

$$\tilde{\gamma}=\gamma_2-e_2, \tag{161}$$

$$X=\frac{b}{h}N+E_0N^2(\frac{b}{h}e_1-\tilde{\gamma}-\frac{b_2}{h})+O(N^3). \tag{162}$$

We do not list the coefficients e_2, γ_2 explicitly since they are too cumbersome. For our purpose, their exact values are irrelevant, it is sufficient to know that they are finite. Then, Eq. (57) obtained from the terms of the order N does not change since one may put there $E=E_0$ directly. Eq. (63) somewhat changes since now the momentum conservation in the second order reads $\tilde{Y}_L=\tilde{Y}_R$ where

$$\tilde{Y}_L=E_0(\frac{b}{h}e_1-\tilde{\gamma})+Y_L \tag{163}$$

and

$$\tilde{Y}_R\equiv -\varepsilon_3\tau. \tag{164}$$

Here, Y_L is obtained from (64) by substitution $E_1\rightarrow E_0$. What is important is that the quantity m_4 is still finite. In the Kerr case $E_0=\frac{m}{\sqrt{3}}$, $e_1=4/3$, the coefficient $\tilde{\gamma}=1$, so Eq. (157) corresponds to Eq. (A1) and Eq. $\tilde{Y}_L=\tilde{Y}_R$ with (163), (164) corresponds to Eq. (A4) of [9].

XIII. APPENDIX B: CONDITION $S \geq 0$

The contents of the present Appendix generalizes Appendix B of [9]. Let us consider the quantity Y_R (65) which represents the right hand side of Eq. (63) that follows from the momentum conservation (20) in the second order with respect to N. We are interested in the case $C_1 = 0$ when $E_3 = \lambda_+$ and (102) reduces to

$$S = S_1 + S_2, \tag{165}$$

$$S_1 = \frac{1}{h^2}(bb_2 - \frac{g_1}{2g_H})[-\varepsilon_3 \frac{\lambda_+^2}{\sqrt{\lambda_+^2[(\frac{b}{h})^2 - \frac{1}{h^2}] - m_3^2)}} - \frac{E_1^2}{\sqrt{E_1^2\frac{(b^2-1)}{h^2} - m_1^2}}], \tag{166}$$

$$S_2 = C_2\lambda_+[\frac{-\varepsilon_3\frac{b}{h}\lambda_+}{\sqrt{\lambda_+^2[(\frac{b}{h})^2 - \frac{1}{h^2}] - m_3^2)}} - 1] \tag{167}$$

Different options should be considered separately.

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$C_2 \geq 0$, $\varepsilon_3 = -1$, so $S_2 \geq 0$. Let, additionally, $bb_2 - \frac{g_1}{2g_H} \geq 0$. Now we will show that also $S_1 \geq 0$. under additional conditions which are not very restrictive. To simplify matter further, we put $m_3 = 0$ and take $E_3 = \lambda_+$ similarly to [9]. We have

Indeed,

$$\frac{\lambda_+^4}{E_3^2\frac{(b^2-1)}{h^2} - m_3^2} - \frac{E_1^4}{E_1^2\frac{(b^2-1)}{h^2} - m_1^2} = \frac{D}{(E_3^2\frac{(b^2-1)}{h^2} - m_3^2)(E_1^2\frac{(b^2-1)}{h^2} - m_1^2)}, \tag{168}$$

$$D = \lambda_+^4[E_1^2\frac{(b^2-1)}{h^2} - m_1^2] - E_1^4\lambda_+^2\frac{(b^2-1)}{h^2}, \tag{169}$$

$$\frac{h^2 D}{E_1^4\lambda_+^2} \equiv f(x) = x^2\frac{\lambda_+^2}{E_1^2} - (b^2 - 1), \tag{170}$$

where

$$x = \sqrt{b^2 - 1 - h^2\frac{m_1^2}{E_1^2}}. \tag{171}$$

It follows from (171) that for $E_1 \geq m_1$

$$\sqrt{b^2 - 1 - h^2} = x_1 \leq x \leq x_2 = \sqrt{b^2 - 1} \tag{172}$$

Assuming $E_1 = m_1$, we obtain from (58) that

$$\frac{A_1}{E_1} = \frac{b - x}{h}. \tag{173}$$

In case $m_3 = 0$, we see from (70) that

$$\frac{\lambda_+}{A_1} = h(b + \sqrt{b^2 - 1}). \tag{174}$$

Then,

$$\frac{\lambda_+}{E_1} = (b + \sqrt{b^2 - 1})(b - x), \tag{175}$$

$$f = (b + \sqrt{b^2 - 1})^2 (b - x)^2 x^2 - (b^2 - 1). \tag{176}$$

It is seen from (171) that

$$f(x_2) = 0. \tag{177}$$

If, additionally, we assume that

$$b^2 > \frac{4(1 + h^2)}{3}, \tag{178}$$

the derivative $\frac{df}{dx} < 0$ in the range (172). In particular, it is satisfied for the Kerr metric. Then, $f \geq 0$, so indeed $S_1 \geq 0$.

If $bb_2 - \frac{g_1}{2g_H} < 0$, the quantity $S_1 < 0$, so the sign of S can be arbitrary.

In scenarios OUT+, OUT- the quantities C_1 and C_2 can be arbitrary, so the sign of S is also arbitrary.

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